THREE-DIMENSIONAL QUASI-STATIONARY ELECTROMAGNETIC FIELD GENERATED BY ARBITRARY CURRENT CONTOUR NEAR CONDUCTING BODY

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The analytical method is developed to calculate the three-dimensional quasi-stationary electromagnetic field generated by arbitrary spatial current contour near the conducting body with plane surface. By the use of displacement currents in dielectric region under quasi-stationary approximation, in addition to the results presented earlier, the solution for the scalar potential and electric intensity in entire dielectric half-space is found. Owing to the established fact of vertical zero components of the electric intensity and current density in conducting half-space, the electric field of surface charge compensates completely the vertical component of induced electric intensity of the initial current system. As an example, the electric intensity and surface electric charge density are calculated for the current contour configuration typical for technological systems. References 13, figures 3.

Key words: analytical method, spatial current contour (closed loop), eddy currents, 3D quasi-stationary electromagnetic field.

Introduction. In many technical applications, the mathematical models used the initial sources of electromagnetic field as alternating currents flowing along the contours of certain configurations are developed to determine the distribution of electromagnetic field [1, 2, 4, 10]. When the initial currents flow near conducting bodies, the eddy currents are induced in them, these currents also form the field [3, 5, 9].

For quasi-stationary electromagnetic field, the problem in general formulation for the case of plane interface between the conducting and dielectric media is considered in [11, 13]. In these works the results attained in [7] are used. In [7] the contours, without loss of generality, are presented as a serial system of emitting dipoles. At the same time, the analytical solution in [11, 13] is found only for magnetic field in the non-conducting region where the current contour is located. The scalar potential and electric intensity in 3D case are left without consideration; the condition of the contour closure is ignored too. Further development of the theory of calculation of the three-dimensional field for arbitrary contours configuration is obtained in [12]. In this paper, under quasi-stationary approximation, the displacement currents are not taken into account in both conducting and dielectric regions. As a result, the analytical solution of the problem for magnetic and electric fields in the conducting region is found. However, the usual restrictions for quasi-stationary problems create difficulties for determination of electric field in dielectric medium in the case of three-dimensional current contours [8, 12].

In this connection, the present paper is aimed at the development of analytical theory for solving the problem of three-dimensional quasi-stationary electromagnetic fields of the current flowing near the conducting magnetizing half-space. That makes it possible to find the solution for both the magnetic and electric field components in the entire space. The main feature of the work consists in the consideration of displacement currents in a dielectric medium.

Mathematical model. The mathematical formulation of the problem in this paper contains both expressions used in [12] and expressions different from them. Therefore, for a clear statement, a complete problem formulation is used in this paper.

By analogy with [12], let us consider the arbitrary contour in a non-conducting medium with relative dielectric permittivity ε_e. The alternating current I_0 flows along the contour located near the conducting body with plane boundary. The body has conductivity γ and relative magnetic permeability μ. The electrophysical parameters within the dielectric and conducting media are not variable in space and in time. The initial current contour is shown in fig. 1, *a* by solid line. As considered, the dimensions of the contour are much less than the dimensions of the plane section of body surface. It gives a possibility to use the model of current contour above the conducting half-space.

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The problem in the terms of the complex amplitudes of electromagnetic field is formulated in the general case as follows. The problem is described by Maxwell equations for the vectors of electric intensity $\boldsymbol{E}$ and magnetic intensity $\boldsymbol{H}$, magnetic flux density $\boldsymbol{B}$ and electrical displacement $\boldsymbol{D}$, total current density $\boldsymbol{j}_0 + \boldsymbol{j}_t = \boldsymbol{j}_0 + \boldsymbol{j} + \boldsymbol{j}_D$ that includes the density of the current from the extraneous sources in the elements of contour $\boldsymbol{j}_0$, conduction current density $\boldsymbol{j}$ and displacement current density $\boldsymbol{j}_D = i\omega\boldsymbol{D}$

$$\begin{cases}\nabla\times\boldsymbol{H} = \boldsymbol{j}_0 + \boldsymbol{j}_t; & \nabla\cdot\boldsymbol{B} = 0;\\ \nabla\times\boldsymbol{E} = -i\omega\boldsymbol{B}; & \nabla\cdot\boldsymbol{D} = 0,\end{cases} \tag{1}$$

where ω is the angular frequency; i is the imaginary unit.

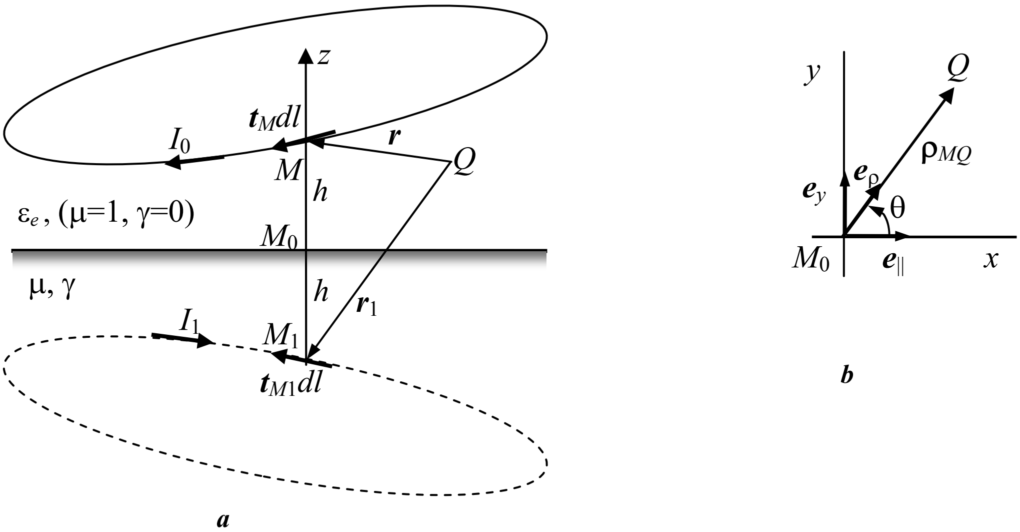

Fig. 1

For the linear problem, the free charge in a piecewise homogeneous medium can be concentrated only on the interface with corresponding surface charge density σ. Then the last equation of system (1) that describes the field out of the boundary surface has no charge.

The initial current contour must be closed for quasi-stationary problem at $I_0 = \text{const}$. In addition, the two conditions are to be satisfied. Firstly, the wavelength of electromagnetic field λ should be much greater than any representative dimension L of the electromagnetic system, i.e. $\lambda = 2\pi / \omega\sqrt{\mu\mu_0\varepsilon\varepsilon_0} >> L$. Secondly, the displacement current density $\boldsymbol{j}_D$ is ignored in a conducting medium in comparison with conduction current density $\boldsymbol{j}$ [6]. Only the displacement current takes place in a dielectric medium. Then the displacement current density is taken into account and the wave phenomena are disregarded.

Using the complex electrical conductivity $\tilde{\gamma} = \gamma + i\omega\varepsilon\varepsilon_0$, in addition to system (1) the constitutive relations are written as

$$\begin{aligned} z > 0: \quad & \mathbf{j}_e = \tilde{\gamma}_e \mathbf{E}_e, \qquad \mathbf{B}_e = \mu_0 \mathbf{H}_e,\\ z < 0: \quad & \mathbf{j}_i = \tilde{\gamma}_i \mathbf{E}_i, \qquad \mathbf{B}_i = \mu\mu_0 \mathbf{H}_i. \end{aligned} \tag{2}$$

Here the subscripts "*e*" and "*i*" correspond to the regions where $z > 0$ and $z < 0$, respectively, $\tilde{\gamma}_e = i\omega\varepsilon_e\varepsilon_0$, $\tilde{\gamma}_i = \gamma$.

The vector and scalar potentials $\boldsymbol{A}$ and φ are introduced

$$\boldsymbol{B} = \nabla\times\boldsymbol{A}; \qquad \boldsymbol{E} = -\nabla\varphi - i\omega\mathbf{A}. \tag{3}$$

The Lorenz gauge condition $i\omega\nabla\cdot\boldsymbol{A} - k^2\varphi = 0$ (where $k^2 = -i\omega\mu\mu_0\tilde{\gamma}$) is used. This condition in the dielectric in contrast to similar expression in [12] as well as in a conducting media has the form

$$\begin{aligned} &in \;\; z > 0: \;\; \nabla\cdot\mathbf{A}_e + i\omega\mu_0\varepsilon_e\varepsilon_0\varphi_e = 0,\\ &in \;\; z < 0: \;\; \nabla\cdot\mathbf{A}_i + \mu\mu_0\gamma\varphi_i = 0. \end{aligned} \tag{4}$$

As a result, the following equations for potentials are obtained from Maxwell equations (1)

$$z>0:\ \Delta \mathbf{A}_e + k_e^2 \mathbf{A}_e = -\mu_0 \mathbf{j}_0, \quad \Delta\varphi_e + k_e^2\varphi_e = 0,$$
$$z<0:\ \Delta \mathbf{A}_i + k_i^2 \mathbf{A}_i = 0, \qquad \Delta\varphi_i + k_i^2\varphi_i = 0. \tag{5}$$

Making an assumption that the conductor is infinitely thin, the current density in (5) is written using the Dirac delta function as $\boldsymbol{j}_0 = I_0\delta(\boldsymbol{r}_M - \boldsymbol{r})\boldsymbol{t}_M$.

The boundary conditions for the tangential and normal components of field vectors are satisfied at the interface of media. Furthermore, no-field condition should be set at infinity

$$\boldsymbol{e}_z \times \left(\boldsymbol{E}^+ - \boldsymbol{E}^-\right) = 0, \qquad \boldsymbol{e}_z \times \left(\boldsymbol{H}^+ - \boldsymbol{H}^-\right) = 0, \tag{6}$$

$$\boldsymbol{e}_z \cdot \left(\boldsymbol{B}^+ - \boldsymbol{B}^-\right) = 0, \qquad \boldsymbol{e}_z \cdot \left(\boldsymbol{j}_t^+ - \boldsymbol{j}_t^-\right) = 0, \tag{7}$$

$$\boldsymbol{A}(\infty) = 0. \tag{8}$$

In going from wave problem for current dipole to quasi-stationary problem, the integration along the closed contour consisting of current dipoles is realized at unvaried complex current amplitude I_0. With a view to exclude wave processes in a dielectric medium at $z>0$, the second summand should be eliminated from the equations in (5).

As follows from (4), if the displacement current is taken into account in a dielectric medium, in order to solve the problem in both conducting and dielectric media it is enough to determine the vector potential distribution.

The normal components of electric intensity on the boundary satisfy the next condition:

$$\left|\boldsymbol{e}_z \cdot E^+\right| / \left|\boldsymbol{e}_z \cdot E^-\right| = \left|\tilde{\gamma}_e\right| / \left|\tilde{\gamma}_i\right| << 1. \tag{9}$$

The terms with factor as a ratio of complex conductivities $\tilde{\gamma}_e/\tilde{\gamma}_i$ should be taken into account only for determination of 3D electric field in a dielectric medium. Really, on the basis of (4), if $\nabla\cdot\boldsymbol{A}_e$ is proportionally with small parameter in (9): $\nabla\cdot\boldsymbol{A}_e \sim \tilde{\gamma}_e/\tilde{\gamma}_i$, then scalar potential has no already such factor – $\varphi_e = i\omega\nabla\cdot\boldsymbol{A}_e/k_e^2 \sim 1/\mu_0\gamma$.

Contrary to that, for a conducting medium, if $\nabla\cdot\boldsymbol{A}_i$ contains the summand with small parameter in (9), then the corresponding summand of scalar potential φ_i has small parameter as a factor. Accordingly, the terms with small parameter (9) can be disregarded for a conducting medium at quasi-stationary problem statement.

The quasi-stationary conjugation problem is formulated on the basis of (9) otherwise, ignoring the displacement current density in a dielectric medium. Under such problem formulation for fully three-dimensional fields the vector potentials $\boldsymbol{A}_e, \boldsymbol{A}_i$ and magnetic flux densities $\boldsymbol{B}_e, \boldsymbol{B}_i$ in all space as well as scalar potentials φ_i and electric intensity $\boldsymbol{E}_i$ in a conducting half-space is found in [12]. In this case the Lorenz gauge condition in a dielectric region is identical with the Coulomb gauge

$$z>0:\ \nabla\cdot\boldsymbol{A}_e = 0. \tag{10}$$

The components of electric intensity which are normal to the boundary are equal to

$$\boldsymbol{e}_z\cdot\boldsymbol{E}^- = 0, \quad \boldsymbol{e}_z\cdot\boldsymbol{E}^+ = \sigma/\varepsilon_e\varepsilon_0. \tag{11}$$

Under this formulation, the scalar potential φ_e in a dielectric medium is undetermined. The scalar potential is not available in the gauge condition (10). The normal electrical field component on plane boundary in (11) is calculated by unknown surface charge density σ. Then using only vector potential, the electric intensity in (3) in region $z>0$ can be evaluated to an accuracy of potential summand. The authors of [8] note this characteristic property of the field conjugation problem in quasi-stationary statement and reveal that the solution is single-valued under complementary conditions.

In this paper, the more general formulation of boundary problem (5)–(8) is used under condition (9) and the transformations valid for closed contour at invariable current flowing along the contour ($I_0 = \text{const}$) are performed at the final stage.

Electromagnetic field in a dielectric region. The analytical solution of 3D problem for harmonic dipole $I_0\boldsymbol{t}dl$ is found in [7] by the complementary two-dimensional Fourier transform in coordinates, the efficiency of which is shown in this paper. The expression for the direct Fourier transform in coordinates for function $f(x,y)$ of two variables has the form

$$f^*(\xi,\eta) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) \cdot e^{-i(\xi x + \eta y)} dx dy \,, \tag{12}$$

where i is the imaginary unit for the given transform.

Using the integral transformation (12) for equations (5), the next one-dimensional equations for vector potential transform are derived instead of three-dimensional equations. The equation for vector potential transform $\boldsymbol{A}_e^*$ in a dielectric region is as follows:

$$\frac{d^2 \boldsymbol{A}_e^*}{dz^2} - q_e^2 \boldsymbol{A}_e^* = -\frac{\mu_0 I_0}{4\pi^2} \delta(z - z_M) \boldsymbol{t}_M dl \,, \tag{13}$$

where $q_e = \xi^2 + \eta^2 - k_e^2 = \vartheta^2 - k_e^2$; $h = z_M = -z_{M1}$ is the vertical coordinate of the source point in the contour. Here the coordinates x, y, z for observation point Q present the local coordinates associated with vertical axis through source point M in the contour. The coordinate x is reckoned in the direction parallel to vector $\boldsymbol{t}_{\|}$, which is the projection of unit tangent vector $\boldsymbol{t}$ to interface surface (fig. 1, *b*).

Taking into account the boundary conditions (6)–(8) expressed in terms of the Fourier transforms of vector potential, the boundary-value problem can be formulated for ordinary inhomogeneous second-order differential equation in coordinate z. Then the solution for Fourier transforms of vector potential is obtained in [7]. Without intermediate calculations, we derive in the form of circulatory integral the solution for Fourier transforms of the vector potential of the serial dipoles distributed along the closed contour at $I_0 = \text{const}$. The result for a dielectric medium (at $z > 0$) is

$$\boldsymbol{A}_e^* = \frac{\mu_0 I_0}{4\pi^2} \oint_l \left\{ \frac{\exp\left(-q_e |z - z_M|\right)}{2q_e} \boldsymbol{t} - \frac{\exp\left(-q_e (z - z_{M1})\right)}{2q_e} \boldsymbol{t}_1 + Y_{e\|}^* \boldsymbol{t}_{\|} + Y_{e1}^* \left(\boldsymbol{t} \cdot \boldsymbol{e}_{\|}\right) \boldsymbol{e}_z + Y_{e2}^* \boldsymbol{t}_{\perp} \right\} dl. \tag{14}$$

Here $\boldsymbol{t}$ and $\boldsymbol{t}_1$ are the unit vectors tangential to the initial contour at point M and to the contour mirrored from interface at point M_1, respectively. The mirror-reflected contour is indicated in fig. 1, *a* by dotted line. The projections of vectors $\boldsymbol{t}$ and $\boldsymbol{t}_1$ on vertical axis are the same in magnitude but opposite in direction: $\boldsymbol{t}_{1\perp} = -\boldsymbol{t}_{\perp}$. Their projections on the interface surface are identical in magnitude and direction: $\boldsymbol{t}_{1\|} = \boldsymbol{t}_{\|}$, i.e. $\boldsymbol{t} = \boldsymbol{t}_{\perp} + \boldsymbol{t}_{\|}$, $\boldsymbol{t}_1 = -\boldsymbol{t}_{\perp} + \boldsymbol{t}_{\|}$. In (14) $\boldsymbol{e}_{\|} = \boldsymbol{t}_{\|} / \left|\boldsymbol{t}_{\|}\right|$ is the unit vector (fig. 1, *b*).

The integration functions in (14) have the following form

$$\begin{aligned} Y_{e\|}^* &= \frac{\mu \exp\left(-q_e (z - z_{M1})\right)}{\mu q_e + q_i}, \\ Y_{e1}^* &= i\xi \frac{\left(\mu \tilde{\gamma}_i - \tilde{\gamma}_e\right) \exp\left(-q_e (z - z_{M1})\right)}{\left(\mu q_e + q_i\right)\left(q_e \tilde{\gamma}_i + q_i \tilde{\gamma}_e\right)}, \\ Y_{e2}^* &= -\frac{q_i \tilde{\gamma}_e \exp\left(-q_e (z - z_{M1})\right)}{q_e \left(q_e \tilde{\gamma}_i + q_i \tilde{\gamma}_e\right)}, \end{aligned} \tag{15}$$

where $q_i^2 = \vartheta^2 - k_i^2$.

If condition (9) is taken into consideration and wave phenomena in a dielectric medium are neglected when $q_e = \vartheta$, then functions in (15) are equal to

$$\begin{aligned} Y_{e\|}^* &= V_{e\|}^* = \frac{\exp\left(-\vartheta (z - z_{M1})\right)}{w(\vartheta)}, \\ Y_{e1}^* &= V_{e1}^* - \frac{\tilde{\gamma}_e}{\tilde{\gamma}_i} P_{e1}^* = i \cos\psi \frac{\exp\left(-\vartheta (z - z_{M1})\right)}{w(\vartheta)} \left(1 - \frac{\tilde{\gamma}_e}{\tilde{\gamma}_i} \frac{\vartheta + \mu q_i}{\mu \vartheta}\right), \\ Y_{e2}^* &= -\frac{\tilde{\gamma}_e}{\tilde{\gamma}_i} P_{e2}^* = -\frac{\tilde{\gamma}_e}{\tilde{\gamma}_i} \frac{q_i \exp\left(-\vartheta (z - z_{M1})\right)}{\vartheta^2}. \end{aligned} \tag{16}$$

where $\cos\psi = \xi / \vartheta$, $w(\vartheta) = \dfrac{\mu \vartheta + q_i}{\mu} = \vartheta + \dfrac{1}{\mu} \sqrt{\vartheta^2 + i\omega \mu \mu_0 \gamma}$.

The vector potential in region $z > 0$ subject to (14)–(16) is expressed by

$$\boldsymbol{A}_e^* = \boldsymbol{A}_{eV}^* + \boldsymbol{A}_{eP}^* = \left(\boldsymbol{A}_0^* + \boldsymbol{A}_1^* + \boldsymbol{A}_2^*\right) + \boldsymbol{A}_{eP}^*, \tag{17}$$

where $\boldsymbol{A}_0^*$ and $\boldsymbol{A}_1^*$ correspond to the first two summands in (14); $\boldsymbol{A}_2^*$ is determined by functions $V_{e\|}^*$ and V_{e1}^* in (16); $\boldsymbol{A}_{eP}^*$ is proportional to small parameter in (9) and determined by functions P_{e1}^* and P_{e2}^* in (16).

In quasi-stationary approximation, the summand $\boldsymbol{A}_{eP}^*$ in the expression for vector potential can be ignored. At the same time, the component $\boldsymbol{A}_{eP}^*$ must be considered to calculate scalar potential φ_e^*.

To derive the solution in terms of physical variables, the inverse Fourier transform in coordinates should be implemented according to the expression

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^*(\xi,\eta)\cdot e^{i(\xi x + \eta y)} d\xi d\eta = \int_0^{\infty} f_1^*(\vartheta)\vartheta d\vartheta \int_{-\pi}^{\pi} f_2^*(\psi) e^{i\vartheta\rho\cos(\psi-\theta)} d\psi, \tag{18}$$

where any component in (15), (16) in the general case is expressed as $f^*(\xi,\eta) = f_1^*(\vartheta) f_2^*(\psi)$.

The result of applying the inverse Fourier transform for all terms in (14)–(16) with the exception of the terms P_{e1}^* and P_{e2}^* which determine summand $\boldsymbol{A}_{eP}^*$ are given in [12]. In this paper the expressions in terms of physical variables for vector potential $\boldsymbol{A}_{eV}$ and magnetic flux density $\boldsymbol{B}_e = \nabla \times \boldsymbol{A}_{eV}$ in a dielectric medium $z > 0$ is found and presented in the following form:

$$\boldsymbol{A}_{eV} = \boldsymbol{A}_0 + \boldsymbol{A}_1 + \boldsymbol{A}_2 = \frac{\mu_0 I_0}{4\pi} \oint_l \left(\frac{\boldsymbol{t}}{r} - \frac{\boldsymbol{t}_1}{r_1} - \boldsymbol{t}_1 \frac{\partial G_e}{\partial z} \right) dl, \tag{19}$$

$$\boldsymbol{B}_e = \boldsymbol{B}_0 + \boldsymbol{B}_1 + \boldsymbol{B}_2 = \frac{\mu_0 I_0}{4\pi} \oint_l \left[\frac{\boldsymbol{t} \times \boldsymbol{r}}{r^3} - \frac{\boldsymbol{t}_1 \times \boldsymbol{r}_1}{r_1^3} - \boldsymbol{t}_1 \times \nabla \left(\frac{\partial G_e}{\partial z} \right) \right] dl. \tag{20}$$

Here subsidiary function G_e is used instead of the sum of two functions $V_{e\|}$ and V_{e1}

$$G_e = 2\int_0^{\infty} \frac{\exp(-\vartheta(z - z_{M1})) J_0(\vartheta\rho)}{w(\vartheta)} d\vartheta. \tag{21}$$

where $J_0(\cdot)$ is the zero-th order Bessel functions of the first kind.

It is easy to check that function G_e in (21) satisfies the Laplace equation $\Delta G_e = 0$. In addition, for any function in the form of $\boldsymbol{t}_{M1} f(\boldsymbol{r}_{QM1})$ the following integral taken along the closed contour is equal to zero

$$\oint_{l_1} \nabla_Q \cdot \left(\mathbf{t}_{M1} f(\mathbf{r}_{QM1})\right) dl_{M1} = -\int_{l_1} \mathbf{t}_{M1} \nabla_{M1} \cdot \left(f(\mathbf{r}_{QM1})\right) dl_{M1} = 0. \tag{22}$$

Consequently, in quasi-stationary approximation, expression (19) for vector potential $\boldsymbol{A}_{eV}$ in a dielectric medium satisfies the continuity condition

$$\nabla \cdot \boldsymbol{A}_{eV} = 0. \tag{23}$$

This result is in agreement with (10) for alternative formulation of conjugation problem. The solution of general three-dimensional problem for magnetic field in quasi-stationary formulation can be represented as three summands, i.e. by field of current contour, field of the current contour mirrored from interface and by the third component to allow for the electrophysical properties of a medium and current frequency.

Below we determine the scalar potential φ_e and electric intensity $\boldsymbol{E}_e$ in region $z > 0$.

On the basis of (4) taking into account (23), the scalar potential is determined by summand of vector potential $\boldsymbol{A}_{eP}$ containing small parameter (9). Furthermore, as seen from (14) and (16), potential $\boldsymbol{A}_{eP}$ has a unique component perpendicular to the interface of media. Then the Fourier transform of scalar potential φ_e^* can be written as

$$\varphi_e^* = \frac{I_0}{4\pi^2\gamma} \oint_l \left[\frac{\partial P_{e1}^*}{\partial z} \left(\boldsymbol{t} \cdot \boldsymbol{e}_{\|}\right) + \frac{\partial P_{e2}^*}{\partial z} \left(\boldsymbol{t} \cdot \boldsymbol{e}_z\right) \right] dl. \tag{24}$$

The inverse Fourier transform (18) gives the following expressions for the functions presented in (24)

$$\frac{\partial P_{e1}}{\partial z} = \frac{2\pi}{\mu}\cos\theta\int_0^\infty \frac{\vartheta + \mu q_i}{w(\vartheta)}\exp\left(-\vartheta(z - z_{M1})\right)J_1(\vartheta\rho)\vartheta d\vartheta ,$$

$$\frac{\partial P_{e2}}{\partial z} = -2\pi\int_0^\infty q_i \exp\left(-\vartheta(z - z_{M1})\right)J_0(\vartheta\rho)d\vartheta . \tag{25}$$

where $J_1(\cdot)$ is the second order Bessel functions of the first kind.

The following subsidiary function G_{e1} is introduced to use the condition of contour closure

$$G_{e1} = \frac{2\pi}{\mu}\int_0^\infty \frac{\vartheta + \mu q_i}{w(\vartheta)}\exp\left(-\vartheta(z - z_{M1})\right)J_0(\vartheta\rho)d\vartheta . \tag{26}$$

For this function $\frac{\partial P_{e1}}{\partial z} = \frac{\partial G_{e1}}{\partial x_{M1}}$. Then the scalar potential is expressed as

$$\varphi_e = \frac{I_0}{4\pi^2\gamma}\oint_l \left\{\left[\frac{\partial G_{e1}}{\partial x_{M1}}\left(\boldsymbol{t}_1 \cdot \boldsymbol{e}_\parallel\right) + \frac{\partial G_{e1}}{\partial z_{M1}}\left(\boldsymbol{t}_1 \cdot \boldsymbol{e}_z\right)\right] - \left(\frac{\partial G_{e1}}{\partial z_{M1}} + \frac{\partial P_{e2}}{\partial z}\right)\left(\boldsymbol{t}_1 \cdot \boldsymbol{e}_z\right)\right\}dl . \tag{27}$$

The last formula takes into account that $\left(\boldsymbol{t} \cdot \boldsymbol{e}_\parallel\right) = \left(\boldsymbol{t}_1 \cdot \boldsymbol{e}_\parallel\right)$ and $\left(\boldsymbol{t} \cdot \boldsymbol{e}_z\right) = -\left(\boldsymbol{t}_1 \cdot \boldsymbol{e}_z\right)$. The integral along the closed contour of the expression in the square brackets in (27) is equal to zero. Then the following expression for scalar potential is derived after some transformations

$$\varphi_e = i\omega\frac{\mu_0 I_0}{4\pi}\oint_l \left(\boldsymbol{t}_1 \cdot \boldsymbol{e}_z\right)G_e dl . \tag{28}$$

The potential component (curl-free component) of electric intensity is written as

$$\boldsymbol{E}_{ep} = -\nabla\varphi_e = -i\omega\frac{\mu_0 I_0}{4\pi}\oint_l \left(\boldsymbol{t}_1 \cdot \boldsymbol{e}_z\right)\nabla G_e dl . \tag{29}$$

As seen, the potential component of electric intensity $\boldsymbol{E}_{ep}$ as well as scalar potential φ_e are equal to zero if only the current contour has no sections perpendicular to the boundary surface.

The vortex (divergence-free) component $\boldsymbol{E}_{eV} = -i\omega\boldsymbol{A}_e$ along with potential component $\boldsymbol{E}_{ep} = -\nabla\varphi_e$ gives the total electric intensity

$$\mathbf{E}_e = \mathbf{E}_{eV} + \mathbf{E}_{ep} = -i\omega\frac{\mu_0 I_0}{4\pi}\oint_l \left\{\frac{\boldsymbol{t}}{r} - \frac{\boldsymbol{t}_1}{r_1} - \left[\boldsymbol{t}_1\frac{\partial G_e}{\partial z} - \left(\mathbf{t}_1 \cdot \mathbf{e}_z\right)\nabla G_e\right]\right\}dl =$$

$$= -i\omega\frac{\mu_0 I_0}{4\pi}\oint_l \left\{\frac{\boldsymbol{t}}{r} - \frac{\boldsymbol{t}_1}{r_1} - \left[\boldsymbol{t}_\parallel\frac{\partial G_e}{\partial z} - \left(\mathbf{t}_1 \cdot \mathbf{e}_z\right)\frac{\partial G_e}{\partial \rho}\mathbf{e}_\rho\right]\right\}dl = -i\omega\frac{\mu_0 I_0}{4\pi}\oint_l \left(\frac{\boldsymbol{t}}{r} - \frac{\boldsymbol{t}_1}{r_1} - \mathbf{e}_z \times \left[\mathbf{t}_1 \times \nabla G_e\right]\right)dl. \tag{30}$$

Note that the third term in the integration function depending on the properties of a conducting medium and field frequency has no component perpendicular to the interface at any configurations and orientations of the initial current contour. It is seen from (30) that for the vortex part of electric intensity, the vertical sections of the contour do not form tangential components. On the contrary, as follows from (30) $\mathbf{E}_{ep}$ contains the tangential components due to flowing initial current in the direction perpendicular to the plane media interface.

Electromagnetic field in a conducting region and conditions on the interface surface. As noted, in quasi-stationary approximation, both the vector and scalar potentials in a conducting medium have no small parameter. In this case, the analytical solution of the problem for magnetic and electric fields in a conducting region is found in [12]. Below the expressions for the vector and scalar potentials and electric intensity are given again to present completely the analytical solution of 3D problem.

The vector and scalar potentials in a conducting half-space are equal to

$$\boldsymbol{A}_i = \frac{\mu_0 I_0}{4\pi}\oint_l \left\{\boldsymbol{t}_\parallel G_{i2} + \boldsymbol{t}_\perp\frac{\partial G_{i1}}{\partial z}\right\}dl , \tag{31}$$

$$\varphi_i = -i\omega\frac{\mu_0 I_0}{4\pi}\oint_l \left(\boldsymbol{t} \cdot \boldsymbol{e}_z\right)G_{i1} dl . \tag{32}$$

where $$G_{i1} = 2\int_0^\infty \frac{\exp(q_i z - \vartheta z_M) J_0(\vartheta\rho)}{w(\vartheta)} d\vartheta, \quad G_{i2} = 2\int_0^\infty \frac{\exp(q_i z - \vartheta z_M) J_0(\vartheta\rho)}{w(\vartheta)} \vartheta d\vartheta. \tag{33}$$

The electric intensity is expressed by the sum of vortex component $\mathbf{E}_{iV} = -i\omega\mathbf{A}_i$ and potential component $\boldsymbol{E}_{ip} = -\nabla\varphi_i$. After some transformations, the expression for electric intensity in a conducting medium takes the form

$$\boldsymbol{E}_i = \boldsymbol{E}_{iV} + \boldsymbol{E}_{ip} = -i\omega\frac{\mu_0 I_0}{4\pi}\oint_l \left\{\boldsymbol{t}_\| G_{i2} - (\boldsymbol{t}\cdot\boldsymbol{e}_z)\frac{\partial G_i}{\partial\rho}\boldsymbol{e}_\rho\right\} dl. \tag{34}$$

Note, here the terms do not correspond of vortex and potential components.

As follows from (34), in the problem for quasi-stationary electromagnetic field at plane interface in a conducting medium the electric intensity and current density have no the components perpendicular to the interface without reference to the configuration of the contour with alternating current. Expression (34) is a consequence of the solution of the wave problem in field theory. At the same time, as shown in [12] the inference for quasi-stationary problem about the null vertical component of electric intensity in a conducting medium has more general reasons. Such conclusion follows from the unique zero solution of the boundary problem of homogeneous second-order differential equation for z-component of electric intensity E_{iz} with zero value on the boundary.

The null vertical components of induced current density in a conducting medium lead to the relations for electric intensity in a dielectric medium $E_z^+(t)$ and for electric charge surface density $\sigma(t)$ at the plane interface. The result is valid for any time variation of current and can be written in the following form:

$$\frac{\sigma(t)}{\varepsilon_e\varepsilon_0} = E_z^+(t) = -2\frac{\partial A_{0z}(z=0,t)}{\partial t}, \tag{35}$$

where

$$-\frac{\partial A_{0z}(z=0,t)}{\partial t} = -\frac{\mu_0}{4\pi}\frac{\partial I_0(t)}{\partial t}\oint_l \frac{(\boldsymbol{t}\cdot\boldsymbol{e}_z)}{r} dl \tag{36}$$

is the vertical component of electric intensity induced by initial alternating current $I_0(t)$.

As follows from superposition principle, the expression (36) remains true for any initial system of closed current contours.

The boundary conditions (6) for the tangential component and condition (7) for the normal component of magnetic flux density are satisfied. This is verified by above presented expressions for the vector potential and electric intensity.

We give an example of the current contour configuration typical for technological systems. Central section of current contour l_1 is located parallel to the plane surface of conducting body (see fig. 2). The sinusoidal current is supplied by two parallel conductors l_2, oriented perpendicularly to the plane of the central section of the contour (fig. 2).

Let us analyze the electric intensity distribution in a dielectric region on the interface of the media ($z = 0$) and find the charge surface density.

The electric intensity for the considered initial spatial contour on the interface in a dielectric half-space has both vertical component E_z^+ and tangential component $\boldsymbol{E}_\|$ (which coincides with the tangential component in a conducting medium). Since at $z = 0$ the distances from the observation point to the source points on the initial and mirror reflected contours are the same ($r = r_1$) and the projections of tangent vectors are equal ($\boldsymbol{t}_{1\|} = \boldsymbol{t}_\|$), the sum of the first two terms in (30) does not create the component parallel to the plane surface.

The tangential component $\boldsymbol{E}_\|$ for the current contour shown in fig. 2 can be presented by sum of two contour integrals along the sections l_1 and l_2

$$\boldsymbol{E}_\| = \boldsymbol{E}_{\|1} + \boldsymbol{E}_{\|2} = i\omega\frac{\mu_0 I_0}{4\pi}\int_{l_1} \boldsymbol{t}_\| \frac{\partial G_e}{\partial z} dl - i\omega\frac{\mu_0 I_0}{4\pi}\int_{l_2} (\boldsymbol{t}_1\cdot\boldsymbol{e}_z)\frac{\partial G_e}{\partial\rho}\boldsymbol{e}_\rho dl. \tag{37}$$

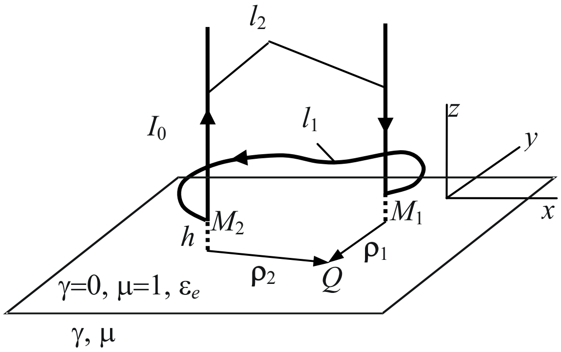

Fig. 2

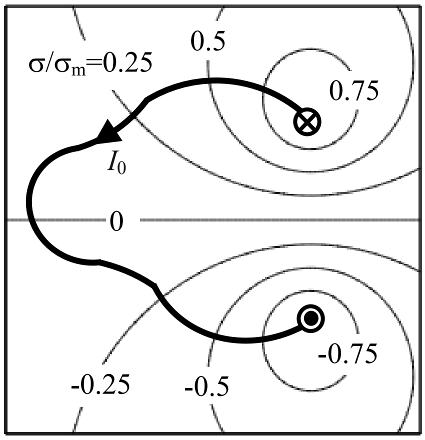

Fig. 3

Note that each separate term in (37) does not satisfy the continuity condition and can not be regarded to separate electric intensity.

As a result of integration along the vertical elements of the contour, the following expression for the term $\boldsymbol{E}_{\|2}$ in (37) is obtained

$$\boldsymbol{E}_{\|2} = i\omega \frac{\mu_0 I}{2\pi} \int_0^\infty \frac{\exp(-\vartheta h)}{w(\vartheta)} \left[J_1(\vartheta\rho_1)\boldsymbol{e}_1 - J_1(\vartheta\rho_2)\boldsymbol{e}_2 \right] d\vartheta , \tag{38}$$

where h is the distance from the central part of the contour to the surface; $\boldsymbol{e}_{1,2} = \rho_{1,2} / |\rho_{1,2}|$.

The value given in (38) must be added to electric intensity $\boldsymbol{E}_{\|1}$ due to current flow through the central horizontal section.

For the example, the surface density of electric charge is found. As the distribution of electric charge is determined only by sections of the contour which are perpendicular to the interface between the media, in this example the charge distribution is the same for any configuration of the central part of the contour.

For the chosen geometry of the contour, the vertical component of vector potential and then the surface charge density can be presented by simple algebraic expression

$$\sigma = -2\frac{i\omega\varepsilon_e\varepsilon_0\mu_0}{4\pi} I_0 \int_h^\infty \left[\left(z_M^2 + \rho_2^2\right)^{-1/2} - \left(z_M^2 + \rho_1^2\right)^{-1/2} \right] dz_M = \frac{i\omega\varepsilon_e\varepsilon_0\mu_0}{2\pi} I_0 \ln \frac{h + \left(h^2 + \rho_2^2\right)^{1/2}}{h + \left(h^2 + \rho_1^2\right)^{1/2}} . \tag{39}$$

Here $\rho_{1,2}^2 = \left(x_Q - x_{1,2}\right)^2 + \left(y_Q - y_{1,2}\right)^2$, where (x_1, y_1) and (x_2, y_2) are the coordinates on the plane surface with two vertical conductors along which the current is directed to the central section of the contour and from it, respectively.

The results of calculation according to (39) are shown in fig. 3 by lines marked as $\sigma/\sigma_m = \text{const}$, where σ_m is the maximum surface density of distributed electric charge.

Conclusion. The analytical solution for three-dimensional quasi-stationary electromagnetic field of alternating current flowing along the contour near the plane surface of conducting magnetizing body is presented without restrictions on the contour configuration, electrophysical properties of media and field frequency. In addition to the results presented earlier, the solution for scalar potential and electric intensity in the entire dielectric half-space is found under quasi-stationary approximation using the displacement currents in dielectric region.

As a revealed peculiarity of the distribution of quasi-stationary electromagnetic field for the systems with plane interface between the dielectric and conducting media, the components of electric intensity and current density which are perpendicular to boundary surface are not available (i.e. equal to zero) in the conducting medium. This result holds true for any spatial distribution of the initial system of non-stationary currents.

As a consequence, at the interface of media the surface density of electric charge and the vertical component of electric intensity in dielectric medium are determined only by the normal component of the induced electric field of initial current system. The electric field of the surface charge completely compensates for the vertical component of electric intensity of induced initial electric field. In this case, the electric charge creates both the vertical component of electric intensity and component parallel to the interface of media.

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ТРИВИМІРНЕ КВАЗІСТАЦІОНАРНЕ ЕЛЕКТРОМАГНІТНЕ ПОЛЕ, СТВОРЕНЕ ДОВІЛЬНИМ КОНТУРОМ СТРУМУ ПОБЛИЗУ ЕЛЕКТРОПРОВІДНОГО ТІЛА

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Розроблено аналітичний метод для розрахунку тривимірного квазістаціонарного електромагнітного поля, створеного струмом, що протікає довільним просторовим контуром поблизу провідного тіла з плоскою поверхнею. Виходячи з використання в квазістаціонарному наближенні струмів зміщення в діелектричній області, на додаток до результатів, представлених раніше, знайдено розв'язок для скалярного потенціалу та напруженості електричного поля в усьому діелектричному півпросторі. Наслідком встановленого факту відсутності компонентів напруженості електричного поля та густини струму в електропровідному середовищі, перпендикулярних до границі поділу середовищ, є повна компенсація поверхневим зарядом вертикальної компоненти індукованої напруженості електричного поля системи вихідного струму. Розглянуто приклад для знаходження напруженості електричного поля та поверхневої густини електричного заряду для конфігурації контуру зі струмом, характерної для технологічних систем. Бібл. 13, рис. 3.

Ключові слова: аналітичний метод, просторовий контур зі струмом, вихрові струми, тривимірне квазістаціонарне електромагнітне поле.

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ТРЕХМЕРНОЕ КВАЗИСТАЦИОНАРНОЕ ЭЛЕКТРОМАГНИТНОЕ ПОЛЕ, СОЗДАННОЕ ПРОИЗВОЛЬНЫМ КОНТУРОМ С ТОКОМ ВБЛИЗИ ЭЛЕКТРОПРОВОДНОГО ТЕЛА

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Разработан аналитический метод для расчета трехмерного квазистационарного электромагнитного поля, созданного током, протекающим по произвольному пространственному контуру вблизи электропроводного тела с плоской поверхностью. Используя в квазистационарном приближении токов смещения в диэлектрической области, в дополнение к результатам, представленным ранее, найдено решение для скалярного потенциала и напряженности электрического поля во всем диэлектрическом полупространстве. Следствием установленного факта отсутствия компонент напряженности электрического поля и плотности тока в электропроводящей среде, перпендикулярных к границе раздела сред, является полная компенсация поверхностным зарядом вертикальной компоненты индуцированной напряженности электрического поля системы исходного тока. Рассмотрен пример для нахождения напряженности электрического поля и поверхностной плотности электрического заряда для конфигурации контура с током, характерной для технологических систем.
Библ. 13, рис. 3.

Ключевые слова: аналитический метод, пространственный контур с током, вихревые токи, трехмерное квазистационарное электромагнитное поле.

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